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# Shallow-water impact problems

ALEXANDER KOROBKIN
*Lavrentyev Institute of Hydrodynamics, Novosibirsk 630090, Russia; e-mail: kaa@hydro.nsc.ru*

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**Abstract.** Impact by a box-like structure onto shallow water is analysed with the help of asymptotic methods. The analysis is based on both the asymptotic approach by Korobkin [1], which was derived originally for blunt-body impact, and the experimental results by Bukreev [2]. The flow region is divided into six parts: the region beneath the entering body, the region close to the bottom edge, the region of inertial flow of the liquid, the jet root, the splash jet and outer region. The flows inside each of the subdomains have their own peculiarities and are analysed separately. The matching conditions make it possible to obtain a uniformly valid asymptotic solution of the impact problem. The main attention is paid to the flow patterns and pressure distributions. It was found that the pressure inside the jet root can be comparable with the pressure beneath the entering body and can even exceed it. The effects of the shape of the body bottom and of the body flexibility on the liquid flow and the pressure distribution are investigated.

**Keywords:** asymptotic methods, shallow water, pressure distribution, spray jets, water impact.

## 1. Introduction

The study of impact onto shallow water is often motivated by its application to the problem of tsunami generation. Under the impact a part of the liquid is piled up and may form a tsunami thereafter due to gravity. In a similar manner as for tsunami generated by an earthquake, the process can be divided into two parts: at the initial stage (impact stage), the duration of which is relatively short, gravity can be neglected and deformation of the liquid domain is mainly due to the body entry; at the second stage (gravity stage) the liquid flow is determined by the gravity, which acts to restore the elevated portion of the liquid, with a possible formation of a solitary wave. Parameters of the solitary wave can be approximately determined if the shape of the liquid elevation at the beginning of the gravity stage is known.

The main focus of the present study is the impact stage, but the pattern of the flow generated by impact can give helpful ideas about a portion of the body energy, which is adopted by tsunami. In particular, it was revealed by Korobkin [1] and confirmed experimentally by Bukreev [2] that blunt-body impact does not lead to soliton formation: most of the liquid displaced by the entering body leaves the liquid layer as a spray jet. The jet is strong: its thickness is comparable with the initial depth of the liquid. This effect can be explained as follow: the possibility of the liquid free surface to move up is restricted by the surface of the entering body; this causes the liquid particles to change the directions of their motion and to leave the main volume tangentially to the entering surface. One may expect that under the impact by a box-like structure, which does not provide any restriction to the free surface elevation, the spray jet is not strong and a solitary wave can be generated.

A sketch of the flow in the plane case is shown in Figure 1. Initially the liquid is at rest and occupies a region $-h < z < 0$. The bottom of a box-like structure touches the liquid over the

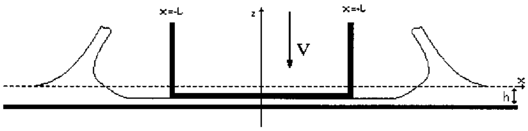


*Figure 1.* Impact of a box-like structure on a thin liquid layer. The flow pattern: the roots of the splash jets move from the body and the jets are inclined towards the body. Initially, the liquid is at rest and occupies the strip $-h < z < 0$, and the body touches the liquid surface over the interval $-L < x < L$. The initial position of the liquid surface is shown by the broken line.

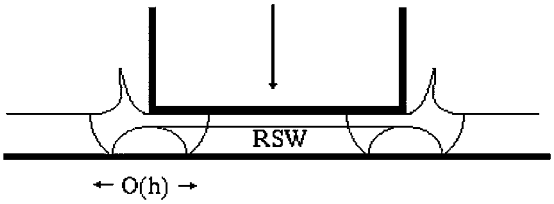


*Figure 2.* The liquid flow just after the impact: RSW, the shock wave reflected from the bottom. Fronts of relief waves, which come from the free surface, are almost circular. Outflow of the liquid from beneath the body is still not developed.

interval $-L \leqslant x \leqslant L, z = 0$. The origin of the Cartesian coordinate system $Oxz$ is taken at the centre point of the interval. At some instant of time, taken as the initial one ($t = 0$), the body begins to penetrate the liquid layer, the initial impact velocity being $V_0$. The position of the body bottom at an instant $t$ is given by $z = -s(t)$, where $s(t)$ is the penetration depth and $s'(0) = V_0$. A prime stands for the time derivative.

We shall determine the liquid flow, its boundary geometry and the pressure distribution up to the moment $T$ of the body contact with the bottom under the following assumptions:

(i) the liquid is ideal and weakly compressible;
(ii) the structure is undeformable;
(iii) the liquid flow is plane, symmetrical with respect to the $z$-axis and irrotational;
(iv) external mass forces and surface tension are absent;
(v) the thickness $h$ of the liquid layer is much smaller than the dimension of the body bottom $L$;
(vi) the Mach number $M = V_0/c_0$, where $c_0$ is the sound velocity in the resting liquid, is small;
(vii) the liquid depth $h$ is much greater than $ML$.

The impact of a box-like structure onto a liquid layer may be divided into the following stages.

(i) At the first stage (see Figure 2), the duration of which is of the order $O(h/c_0)$, compressibility effects are of major importance. Outflow of the liquid from beneath the body is not developed. The pressure beneath the body grows in time. The horizontal component of the liquid velocity only differs from zero close to the edges of the body bottom. The dimension of the edge vicinity is of $O(h)$. The flow inside the vicinity is two-dimensional. The spray jet is formed just after the moment of impact. Its distance from the edge is of $O(h)$.

(ii) At the second stage (see Figure 3), the duration of which is of the order $O(L/c_0)$, the outflow from beneath the entering body will develop. The pressure beneath the body starts

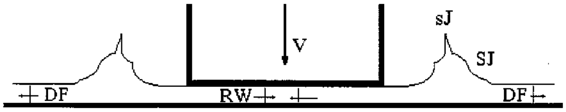


*Figure 3.* The liquid flow at the second stage. The liquid outflow from beneath the body is under development, and the splash jet is under formation. The spray jet still exists but does not give an important contribution to the flow: sJ, the spray jet; SJ, the splash jet; DF, disturbance fronts; RW, relief wave. The figure corresponds to an instant of time when the horizontal velocity of the liquid is zero near the centre line, $x = 0$.

to drop owing to relief waves coming from the free surface. Assumption (vii) provides the estimate $V_0 L/(hc_0) \ll 1$, which means that we can neglect the body displacement during this stage. At the end of this stage acoustic effects can be disregarded and the flow is almost incompressible and linear. The splash jet is formed owing to the liquid outflow from beneath the body, the root of the jet is near the edges of the body bottom. The spray jet is still visible at the end of the splash jet but it does not give a significant contribution to the main flow.

(iii) At the third stage, the duration of which is of the order $O(h/V_0)$, the outflow of liquid from beneath the body has already developed, the flow is unsteady and nonlinear. The body displacement is comparable with the liquid layer thickness. Hydrodynamic loads on the body are growing again owing to the decrease of the gap between the body and the bottom. The root of the splash jet moves from the body, its speed is determined by the outflow velocity. Between the body and the jet root, the region of inertial liquid flow appears. This is the stage which is being considered in the present paper. It is clear that the durations of both the first and second stages are much smaller than the duration of the third stage.

The plane problem on the penetration of a box-like structure at a given velocity is analysed first in Sections 3–7. The effect of the body flexibility and three-dimensional effects are studied in Sections 8 and 9, respectively.

## 2. Formulation of the problem

The entry of a box-like structure into a layer of ideal and weakly compressible liquid is considered (see Figure 1). The structure is rigid and the flow is plane and irrotational. At the initial instant of time ($t = 0$), the body touches the liquid boundary over the interval $-L < x < L$, $z = 0$, the centre point of the interval is taken as the origin of the Cartesian coordinate system $Oxz$. The liquid initially is at rest and occupies the strip $-h < z < 0$. The line $z = 0$ corresponds to the undisturbed position of the liquid boundary. The part of the boundary $-L < x < L$, $z = 0$ corresponds to the contact region of the structure bottom with the liquid, and the parts $x < -L$ and $x > L$ correspond to the free surface where the pressure is zero at all times. The line $z = -h$ corresponds to the rigid bottom of the liquid. Next the body starts to penetrate the liquid layer at an initial impact velocity $V_0$. The position of the body bottom is given by the equation $z = -s(t)$, where $s(t)$ is the depth of penetration. It is necessary to determine the liquid flow and the pressure distribution under the assumptions (i)–(vii) of the previous section.

The liquid motion is governed by the Euler equations

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x}, \tag{1}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g \tag{2}$$

with respect to the velocity vector of liquid particles $\mathbf{u}(x, z, t) = (u, v)$, the pressure $p(x, z, t)$ and the liquid density $\rho(x, z, t)$, where $\rho(x, z, 0) = \rho_0$, and $g$ is the acceleration due to gravity. The equation of continuity for a compressible liquid is

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial z}(\rho v) = 0. \tag{3}$$

The equation of state is taken in the Tate form and can be written as (see [3])

$$\rho = \rho_0(1 + np/\rho_0 c^2)^{1/n}. \tag{4}$$

(where $n$ = 7·14 for water). The liquid flow is assumed irrotational, therefore, the equation

$$\frac{\partial u}{\partial z} = \frac{\partial v}{\partial x} \tag{5}$$

has to be satisfied. Let $z = \eta(x, t)$ describe the position of the free surface. On the free surface, position of which is unknown in advance, the kinematic condition

$$\frac{\partial \eta}{\partial t} + u\frac{\partial \eta}{\partial x} = v \tag{6}$$

and the dynamic condition

$$p = 0 \tag{7}$$

hold. On the bottom of the entering structure, the normal component of the velocity of the body and that of the liquid particles are equal:

$$v = -s'(t) \qquad (z = -s(t), \mid x \mid < L). \tag{8}$$

On the bottom, $z = -h$, the vertical component of the liquid velocity $v(x, -h, t)$ is zero

$$v = 0 \quad (z = -h, -\infty < x < +\infty). \tag{9}$$

The initial conditions are

$$\mathbf{u} = 0, \quad p = 0, \quad \rho = \rho_0, \quad s = 0, \quad s' = V_0, \quad \eta(x, 0) = 0 \quad (t = 0). \tag{10}$$

For a weakly compressible liquid the Mach number $M = V_0/c_0$ is much less than unity and the durations of both the first and second stages (acoustic stages) are much smaller than the duration of the third stage, at which the depth of the body penetration is comparable with the thickness of the liquid layer. At the third stage the liquid can be treated approximately as incompressible with constant density $\rho_0$. Equation (3) gives

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial z} = 0, \tag{11}$$

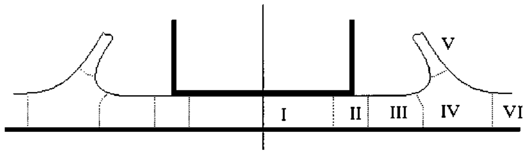


*Figure 4.* Sketch of the liquid flow (spray jets, which are attached to the tops of the splash jets, are not shown): I, the region beneath the entering body; II, the region close to the body bottom edge; III, the region of inertial flow; IV, the jet root; V, the splash jet; VI, the outer region.

the equation of state (4) can be omitted and the initial conditions (10) have to be replaced by matching conditions of the solution at the third stage with the solution at the second stage. Other equations and conditions do not change their forms. There is a small parameter $\varepsilon = h/L$ in the problem (1)–(10). We shall determine an approximate solution of the problem (1)–(10) at the third stage, which is uniformly valid as $\varepsilon \to 0$ up to the moment of contact between the entering body and the bottom.

In order to construct an approximate solution at the third stage, the method of matched asymptotic expansions is used. In accordance with this method the flow is divided into the six regions shown in Figure 4: I, the region beneath the entering body; II, the region close to the bottom edges; III, the region of inertial flow; IV, the jet root; V, the splash jet; VI, the outer region.

In region I, the orders of the independent variables and the unknown functions are as follows: $x = O(L)$, $z = O(h)$, $t = O(h/V_0)$, $v = O(V_0)$, (11) gives that $u = O(V_0 L/h)$, (1) gives $p = O(\rho_0 V_0^2 \varepsilon^{-2})$, (4) gives $\rho = \rho_0(1 + O(M^2\varepsilon^{-2}))$ where $M^2\varepsilon^{-2} \ll 1$ according to assumption (vii) and (5) gives $\partial u/\partial z = O(V_0/L)$. The term $v\partial u/\partial z$ in (1) is much smaller than other terms in the equation, and it may be omitted at leading order as $\varepsilon \to 0$ with relative accuracy of $O(\varepsilon^2)$. Accordingly, all terms in (2) are much smaller than the term $(1/\rho)(\partial p/\partial z)$. This means that in this region $\partial p/\partial z = O(\varepsilon^2[1 + gh/V_0^2])$ and $\partial p/\partial z = 0$ to leading order. The terms with derivatives of the density in (3) are smaller than terms $\partial u/\partial x$ and $\partial v/\partial z$, and they may be omitted at leading order as $M\varepsilon^{-1} \to 0$. This means that the acoustic effects can be neglected with accuracy up to $O(M^2\varepsilon^{-2})$, and Equation (3) can be replaced approximately by (11), which is the equation of continuity for an incompressible liquid. Equation (5) gives at leading order $\partial u/\partial z = 0$. These estimates make it possible to consider the liquid as incompressible and both the pressure $p$ and the horizontal component of the velocity $u$ as approximately $z$-independent. Equations (8) and (9) do not change their forms as $\varepsilon \to 0$.

The flow between two plates, one of which is at rest and another one falling vertically onto it, was studied by Yih [4]. The author was concerned with the infrequency of breakage of glass plates colliding in this way. It was revealed that the presence of air is of major importance at the end of the fall, where the distance between the plates is small.

In region II, the order of the horizontal coordinate $x$ is $O(h)$ but the orders of other variables are the same as in region I. This implies that the flow in this region is quasi-stationary at leading order, and the pressure is zero. The thickness of the liquid layer decreases in time and is equal to $h - s(t)$. The velocity of the liquid particles in the vertical direction, $v$ can be neglected compared with their horizontal velocity $u$. This fact follows from the condition of matching the solutions in regions I and II. Equation (9) gives that the horizontal velocity $u$ does not depend on $x$ at leading order.

In region III, the orders of the variables are the same as in region I. The pressure is zero at leading order as it follows from the dynamic boundary condition (7). Therefore, the liquid particles move inertially. The flow is nonstationary and approximately one-dimensional. It depends mainly on the outflow velocity and the thickness of region I, $h-s(t)$, which provide the boundary conditions at $x=\pm L$. Deformation of the free surface is of $O(h)$. The horizontal dimension of the region can be estimated as the product of the horizontal velocity scale $V_0L/h$ and the timescale $h/V_0$, which is $L$.

The peculiarity of the problem considered is that the flow scheme has to be presented in advance in a way which allow us to construct the approximate and uniformly valid solution. This is possible if the flow in the region IV is essentially two-dimensional. This region, dimension of which is of $O(h)$, moves away from the body at a velocity $c'(t)$, which has to be determined together with the liquid flow. The prime stands for the time derivative. The internal variables $x_1, z_1$ are introduced in this region, $x=c(t)+x_1$, $z=z_1$, where $x_1=O(h)$, $z_1=O(h)$. In order to match the horizontal velocities in regions III and IV, we take $u=O(V_0L/h)$. The equation of continuity (3) gives that the vertical velocity in the region is of the same order, $v=O(V_0L/h)$. The time and the pressure are of the same orders as in region I. The unknown function $c(t)$ is of the order of the dimension of region III, that gives $c(t)=O(L)$ and $c'(t)=O(V_0L/h)$. This means that region IV propagates at a velocity which is comparable with the velocity of the liquid outflow from beneath the entering body. The derivative in time $\partial/\partial t$ is transformed in the internal variables into the operator $\partial/\partial t - c'(t)\partial/\partial x_1$ which in dimensionless variables has the form

$$\frac{V_0}{h}\left(\frac{\partial}{\partial(tV_0/h)}-\frac{c'(t)}{V_0}\frac{\partial}{\partial(x_1/h)}\right).$$

Here $c'(t)/V_0=O(\varepsilon^{-1})$ and is much greater than unity. This indicates that derivatives in time, $\partial/\partial t$, in the original formulation of the problem can be substituted by $-c'(t)\partial/\partial x_1$ with accuracy up to $O(\varepsilon)$. This means that the flow in region IV may be approximately considered as quasi-stationary at leading order as $\varepsilon\to 0$.

In the jet region V the pressure is near the atmospheric value and so the liquid particles move inertially. The flow in region III depends on that in the jet root; the influence of the jet flow on the flows in others regions may be neglected.

In region VI the liquid remains at rest. This is possible if the speed of region IV propagation, $c'(t)$, is greater than the critical velocity for the liquid layer, which is $(gh)^{1/2}$. The condition is satisfied if $V_0L/h \gg (gh)^{1/2}$. When $h=10\,\text{cm}$, $L=50\,\text{cm}$ the inequality gives $V_0 \gg 20\,\text{cm}\,\text{s}^{-1}$.

We shall determine both the flows and the pressure distributions in region I, III, IV, and match them with each other and with the rest state in region VI. The first-order solution in region II is trivial and does not provide additional matching conditions. The shape of the jet region can be determined after the flow in region IV is found. This procedure makes it possible to find approximately all characteristics of the liquid flow during the third stage of the body impact onto shallow water.

It is important that simplified models, which approximately describe the liquid flows in regions I–VI, do not require initial conditions. This makes it possible to study the peculiarities of the flow at the third stage separately and independently on the analysis of the previous stages.

## 3. Liquid flow and pressure distribution beneath the entering body

The asymptotic analysis and physical reasonings (see previous section) indicate that the flow in region I can be described within the framework of the shallow water theory. The flow is governed approximately by the equations

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -\frac{1}{\rho_0}\frac{\partial p}{\partial x}, \tag{12}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial z} = 0 \quad (\mid x \mid < L,\ -h < z < -s(t)), \tag{13}$$

where $u = u(x,t)$, $p = p(x,t)$, $v = v(x,z,t)$. The boundary conditions (8) and (9) give

$$v = -s'(t) \quad (z = -s(t), \mid x \mid < L), \tag{14}$$

$$v = 0 \quad (z = -h, \mid x \mid < L). \tag{15}$$

Matching the pressures in regions I and III, we obtain (see also [4])

$$p = 0 \quad (x = \pm L). \tag{16}$$

The flow is symmetrical, which gives

$$u = 0 \quad (x = 0,\ -h < z < -s(t)). \tag{17}$$

Integrating (13) with respect to $z$ and taking (14), (15) and (17) into account, we obtain the velocity field of the flow at leading order

$$u(x,t) = \frac{s'(t)x}{h - s(t)}. \tag{18}$$

Equation (12) and the boundary condition (16) provide the pressure distribution beneath the body (see also [4])

$$p(x,t) = p(0,t)\left(1 - \frac{x^2}{L^2}\right), \tag{19}$$

$$p(0,t) = \tfrac{1}{2}\rho_0 L^2\left[\left(\frac{s'}{h-s}\right)^2 + \left(\frac{s'}{h-s}\right)'\right]. \tag{20}$$

The hydrodynamic force $F(t)$ on the entering body is

$$F(t) = \tfrac{4}{3}Lp(0,t). \tag{21}$$

In particular, if $s(t) = V_0 t$, we obtain

$$p(0,t) = \rho_0 V_0^2 \varepsilon^{-2}(1 - V_0 t/h)^{-2},$$

which indicates that the hydrodynamics loads grow beyond all bounds when the body approaches the bottom.

The flow in region I does not depend at leading order on peculiarities of the liquid motion in other regions and can be determined separately. If the loads on the entering body are of major importance, Equations (18)–(21) provide all necessary quantities. The flow outside region I is determined at leading order by the outflow velocity $u(L,t)$ only.

## 4. Inertial outflow

The flow in region III is governed by the equations

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0, \tag{22}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial z} = 0 \qquad (L < x < c(t),\ -h < z < \eta(x,t)), \tag{23}$$

the kinematic condition on the free surface (6) gives

$$\frac{\partial \eta}{\partial t} + u\frac{\partial \eta}{\partial x} = v, \qquad (L < x < c(t),\ z = \eta(x,t)), \tag{24}$$

the boundary condition on the bottom (9) provides

$$v = 0 \quad (L < x < c(t),\ z = -h). \tag{25}$$

Matching the horizontal velocities and the thicknesses of the layers in regions I and III close to the body edge, $x = L$, leads to the boundary conditions

$$u(L,t) = \frac{s'(t)L}{h - s(t)}, \tag{26}$$

$$\eta(L,t) = -s(t), \tag{27}$$

as it follows from Equations (18) and (14). The solution of Equation (22), which satisfies the boundary condition (26), in general case is complicated. We restrict ourselves by the case of constant velocity of the body penetration, $s(t) = V_0 t$. In this case the solution of problem (22) and (26) is

$$u(x,t) = \frac{V_0(2L - x)}{h - V_0 t} \quad (L < x < c(t)). \tag{28}$$

Equations (23)–(25), (27) and (28) lead to the equation for the free surface evolution $\eta(x,t)$

$$\eta_t + \frac{V_0(2L - x)}{h - V_0 t}\eta_x = \frac{V_0}{h - V_0 t}(h + \eta) \qquad (L < x < c(t), t > 0), \tag{29}$$

and the boundary condition

$$\eta(L,t) = -V_0 t. \tag{30}$$

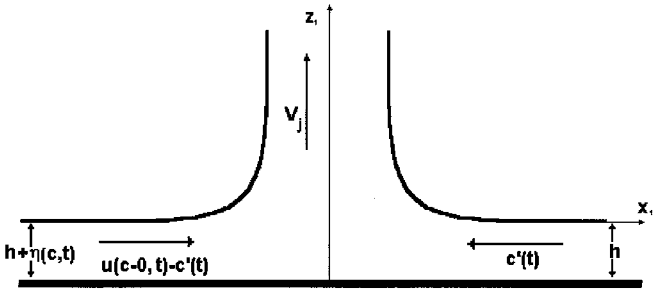


*Figure 5.* The flow pattern in the jet root region shown within the moving coordinate system.

The solution of the problem (29) and (30) is

$$\eta(x,t) = L^2 \frac{h - V_0 t}{(2L - x)^2} - h. \qquad (31)$$

Formula (31) shows that

$$\eta_x(L,t) = 2\varepsilon(1 - V_0 t/h).$$

This means that the derivatives $p_x$, $u_x$ and the slope of the liquid boundary are not continuous at $x = L$. In order to match them, high order solutions in region II are required. The function $c(t)$ is still undetermined. In order to find it, the flow in region IV is considered.

## 5. Jet root

In the moving coordinate system which translates away from the body at the velocity $c'(t)$, the flow in region IV can be approximately considered as quasi-stationary (see Figure 5). Within the framework of this scheme the flow in this region is similar to the flow due to the collision of two plane jets. Parameters of the jets follow from the matching conditions of the flow in region IV with the flows in regions III and VI: the jet of the thickness $h$ moves left at the velocity $c'(t)$ and the jet of the thickness $h + \eta(c,t)$ moves right at the velocity $u(c,t) - c'(t)$. As a result of the collision the splash jet with the thickness $h_j$ is formed, the jet velocity being $V_j$. The quantities $h_j$ and $V_j$ are unknown in advance. The dynamical condition on the free surface demands that the magnitude of the flow velocity on the free surface in the quasi-stationary case is constant. This gives $V_j = c'(t)$ and

$$u(c(t),t) - c'(t) = c'(t). \qquad (32)$$

Equation (32) together with the initial condition

$$c(0) = L \qquad (33)$$

form the initial- value problem

$$\frac{\mathrm{d}c}{\mathrm{d}t} = \frac{V_0}{2} \cdot \frac{2L - c}{h - V_0 t} \quad (t > 0), \quad c(0) = L \qquad (34)$$

with respect to the function $c(t)$. We find

$$c(t) = 2L - L\sqrt{1 - V_0 t/h}, \tag{35}$$

$$c'(t) = \frac{1}{2}\frac{V_0 L}{h}(1 - V_0 t/h)^{-1/2}. \tag{36}$$

Equation (31) provides $\eta(c(t), t) = 0$. This implies that two plane jets of the same thickness $h$ collide at the velocity $c'(t)$ given by (36) and the splash jet moves vertically at the same velocity, $V_j(t) = c'(t)$. The thickness of the spray jet $h_j$ can be found from the mass conservation law. We obtain that $h_j$ is equal to $2h$ and does not change its value with time. The pressure at the stagnation point is given approximately by the Bernoulli's equation $p_{st}(t) = \frac{1}{2}\rho_0[c'(t)]^2$. In the case of the constant velocity of the body penetration, we obtain from (36)

$$p_{st}(t) = \tfrac{1}{8}\rho_0 V_0^2 \varepsilon^{-2}(1 - V_0 t/h)^{-1}. \tag{37}$$

In particular,

$$\frac{p_{st}(t)}{p(0,t)} = \frac{1}{8}\left(1 - \frac{V_0 t}{h}\right).$$

This means that the pressure in the jet roots, which translate away from the body at the velocity $c'(t)$ given by (36), is comparable with the pressure beneath the entering body, but their ratio vanishes in time. We may expect that more complicated laws of the body motion lead to different rations between the maximum pressures in regions I and IV.

It is important to notice that $\eta_x(c,t) = 2\varepsilon(1 - V_0 t/h)^{-1/2}$, therefore $\eta_x(c,t) \to \infty$ and $c'(t) \to \infty$ as $V_0 t/h \to 1$, and the approximate solutions in regions I–IV are not valid at the end of the third stage when $h - V_0 t \to 0$. In order to describe correctly the flow up to the moment of contact, $t = h/V_0$, compressibility of the liquid has to be taken into account. On the other hand, the experiments by Bukreev [2] indicate that velocity of the body penetration is not constant, it rapidly decays as the body approaches the bottom of the liquid layer. Variation of the body velocity owing to the interaction of the entering body with the liquid has to be taken into account.

## 6. Splash jet

In the jet region the pressure is near the atmospheric value and the liquid particles move inertially, both the vertical and horizontal components of their velocity being equal to $c'(t)$. Equations of liquid motion predict that within the framework of the original coordinate system $xOz$ the position of the jet and its velocity are given in parametric forms as

$$x = c'(\tau)(t - \tau) + c(\tau), \quad u = c'(\tau), \tag{38}$$

$$z = c'(\tau)(t - \tau), \quad v = c'(\tau),$$

where $\tau$ is the parameter, $0 \leqslant \tau \leqslant t$. The shape of the jet is hyperbolic

$$\left(\frac{2L - x}{L\sqrt{1 - V_0 t/h}}\right)^2 - \left(\frac{z}{L\sqrt{1 - V_0 t/h}}\right)^2 = 1, \tag{39}$$

where $L + V_0 Lt/2h < x < c(t)$, if the entry velocity of the body is constant. Equation (39) follows from (35), (36) and (38) and predicts that far from the jet root the splash jet is inclined towards the body at angle $\pi/4$ but propagates away from the body.

In order to determine the jet thickness, we introduce two functions $S(z,t)$ and $\zeta(z,t)$ which describe the shape of the left-hand free surface of the jet

$$x = S(z,t)$$

and the shape of its right-hand free surface

$$x = S(z,t) + h\zeta(z,t).$$

The kinematic boundary condition provides

$$\frac{\partial}{\partial t}S + v\frac{\partial}{\partial z}S = u \qquad (x = S(z,t), z > 0, t > 0), \tag{40}$$

$$\frac{\partial}{\partial t}(S + h\zeta) + v\frac{\partial}{\partial z}(S + h\zeta) = u \qquad (x = S(z,t) + h\zeta(z,t), z > 0, t > 0). \tag{41}$$

Taking (38) into account, we can rewrite Equation (40) in the form

$$\frac{\mathrm{d}S}{\mathrm{d}t} = u \quad (t > 0), \quad S(0,\tau) = c(\tau), \tag{42}$$

where $\mathrm{d}/\mathrm{d}t = \partial/\partial t + v\partial/\partial z$. The solution of problem (42) is

$$S = c'(\tau)(t - \tau) + c(\tau),$$

where $\tau = \tau(z,t)$ is determined by the equation

$$z = c'(\tau)(t - \tau).$$

Equation (41) with account for (40), (5) and (11) yields at leading order as $\varepsilon \to 0$

$$\frac{\mathrm{d}}{\mathrm{d}t}(\log\zeta) = [v_z + S_z \cdot u_z]_{x=S(z,t)}. \tag{43}$$

The parametric forms of the velocity components (38) provide

$$v_z = c''(\tau)\frac{\partial\tau}{\partial z}, \quad u_z = c''(\tau)\frac{\partial\tau}{\partial z}, \quad S_z = c''(\tau)(t-\tau)\frac{\partial\tau}{\partial z},$$

$$c''(\tau)\frac{\partial\tau}{\partial z} = \frac{1}{t - B(\tau)}, \quad B(\tau) = \tau + c'(\tau)/c''(\tau).$$

By algebra

$$[v_z + S_z \cdot u_z]_{x=S(z,t)} = \frac{2}{t-B} + \frac{B-\tau}{(t-B)^2}. \tag{44}$$

Matching the jet thickness in regions IV and V provides the initial condition

$$\zeta(0, t) = 2. \tag{45}$$

The solution of problem (43)–(45) is

$$\zeta_1(t, \tau) = 2\left(\frac{\tau - B(\tau)}{t - B(\tau)}\right)^2 \exp\left(\frac{t - \tau}{t - B(\tau)}\right), \tag{46}$$

where $\zeta(z, t) = \zeta_1(t, \tau(z, t))$. Equations (38) and (46) determine the jet thickness in parametrical form.

In the case $s(t) = V_0 t$, Equation (46) provides

$$\zeta(z, t) = 2eA^2(z, t)\mathrm{e}^{-A(z,t)},$$

$$A(z, t) = 1 + \frac{z_2\left(z_2 + \sqrt{z_2^2 + 1 - t_2}\right)}{1 - t_2 + z_2\left(z_2 + \sqrt{z_2^2 + 1 - t_2}\right)}, \tag{47}$$

$$z_2 = z/L, \qquad t_2 = V_0 t/h.$$

It is worth noting that the jet is inclined at angle 45° towards the body but moves away from the body as $t_2 \to 1$; this follows from (39), with $S(z, 1) = 2L - z$ and the jet thickness differs from constant, $\zeta(z, 1) \equiv 8/\mathrm{e}$ where $z > 0$, in the jet root region only. More details on a splash from a thin layer of water were given by Peregrine [5].

## 7. Pressure distribution

When the penetration velocity is constant the pressure $p_{st}(t)$ in the jet root is smaller than the pressure at the centre point of the body $p(0, t)$. The experiments by Bukreev [2] revealed that the entry velocity drops essentially owing to the interaction of the body with the liquid and vanishes as the body approaches the bottom. It is expected that the pressure distribution in this case will be different from that obtained in Section 5.

In order to demonstrate this point, consider the outflow velocity $\hat{u}(t) = u(L, t)$ given as $\hat{u}(t) = u_0 - \alpha t$, which corresponds to the body motion

$$s(t) = h\left[1 - \exp\left(-\frac{u_0 t}{L} + \frac{\alpha t^2}{2L}\right)\right].$$

The body velocity $s'(t)$ is positive

$$s'(t) = \frac{h}{L}\hat{u}(t)\exp\left(-\frac{u_0 t}{L} + \frac{\alpha t^2}{2L}\right)$$

when $t < u_0/\alpha$. Both the position and the velocity of the jet root are determined by the equation $\mathrm{d}c/\mathrm{d}t = u(c, t)/2$ and can be found in parametric form as

$$c = (u_0 - \alpha\tau)(t - \tau) + L,$$

$$c' = \tfrac{1}{2}(u_0 - \alpha\tau),$$

$$t = \tau + \int_0^\tau \frac{\hat{u}^2(\nu)}{\hat{u}^2(\tau)} \mathrm{d}\nu.$$

The equations provide

$$32(c')^3 - 12(c')^2\hat{u}(t) = u_0^3. \tag{48}$$

It is convenient to introduce the new variable $k(t)$ such that $c'(t) = \hat{u}(t) \cdot k$. Equation (48) gives

$$32k^3 - 12k^2 = \left[\frac{u_0}{\hat{u}(t)}\right]^3. \tag{49}$$

The pressure $p(0, t)$ at the centre of the body bottom is

$$p(0, t) = \tfrac{1}{2}\rho_0(\hat{u}_t(t) \cdot L + \hat{u}^2(t))$$

and, therefore

$$\frac{p_{st}(t)}{p(0, t)} = \frac{(c')^2(t)}{L\hat{u}_t(t) + \hat{u}^2(t)} = \frac{k^2}{1 - \alpha L/\hat{u}^2(t)}.$$

This ratio is greater than unity if $k > 1$. The right-hand side of Equation (49) grows in time. This means that the solution of this equation is greater than unity starting from the instant $t_*$ when $\hat{u}(t_*) = u_0 \times 20^{-1/3}$. We obtain $t_* = u_0(1 - 20^{-1/3})/\alpha$. Therefore, the pressure at the jet root can be higher than the pressure beneath the body.

It is of interest that $p(0, t)$ is negative when $t > (u_0 - \sqrt{\alpha L})/\alpha$. This means that cavitation may occur near the centre line beneath the entering body.

## 8. Impact by a body with elastic bottom

It was shown in the previous sections that at leading order as $\varepsilon \to 0$ the flow beneath the entering body (region I) can be considered independently of flows in other regions. On the other hand, the flow outside of region I is determined by the outflow velocity $\hat{u}(t) = u(L, t)$ and by the thickness of the region at the edges, $x = \pm L$. Details of the flow between the entering body and the liquid bottom are not very important for peculiarities of the liquid motion in regions III–V.

Consider the plane and unsteady problem of impact onto shallow water by a structure with an elastic bottom. We assume that the bottom deflection is governed by the Euler beam equation

$$mw_{tt} + EJw_{xxxx} = p(x, t) \quad (|\, x \,| < L), \tag{50}$$

where $m$ is the beam mass per unit length, $E$ is the modulus of elasticity and $J$ is the inertia momentum of the beam cross-section. The ends of the beam are assumed simply supported hence

$$w = w_{xx} = 0 \quad (x = \pm L). \tag{51}$$

Deformations and velocities obtained by the beam at the acoustic stages are assumed small compared with the depth of the liquid layer $h$ and the penetration velocity $V_0$, respectively. This leads to the initial conditions

$$w = w_t = 0 \quad (t = 0). \tag{52}$$

Condition (8) has to be replaced for a flexible bottom by

$$v = -s'(t) + w_t(x,t) \qquad (z = -s(t) + w(x,t), \mid x \mid < L). \tag{53}$$

Equations (1)–(7) and (9), (10) remain valid in the case of elastic body impact.

At leading order as $\varepsilon \to 0$ we obtain, in the same manner as in Section 3, the following two equations

$$u_t + uu_x = -\frac{1}{\rho_0} p_x, \tag{54}$$

$$w_t = V_0 - \frac{\partial}{\partial x}[u(x,t)(h - V_0 t + w(x,t))], \tag{55}$$

which together with (50)–(52) and the boundary conditions

$$p(L,t) = 0, \qquad u(0,t) = 0 \tag{56}$$

provide the initial value problem with respect to the horizontal velocity $u(x,t)$, the pressure $p(x,t)$ and the bottom deflection $w(x,t)$. The problem is unsteady and nonlinear.

The orders of the independent variables and the unknown functions in region I are the same as for an undeformable bottom. Moreover, it is reasonable to assume that $w = O(h)$. In this case the relative orders of the terms in (50) are

$$\frac{mw_{tt}}{p(x,t)} = O\left(\frac{m}{\rho_0 L}\varepsilon\right), \qquad \frac{EJw_{xxxx}}{p(x,t)} = O\left(\frac{EJ}{\rho_0 V_0^2 L^3}\varepsilon^3\right).$$

For a beam of constant thickness $h_b$ we have $m = h_b\rho_b$, where $\rho_b$ is the density of the beam. The ratio $m\varepsilon/\rho_0 L$ is equal to $(\rho_b/\rho_0)(h_b/L)\varepsilon$ and is small. The second quantity, $EJ\varepsilon^3/(\rho_0 V_0^2 L^3)$, can be large if the penetration velocity $V_0$ is small. For example, for the steel plate used in the drop tests carried out at MARINTEX [6] and the liquid depth $h = 3$ cm, we obtain $V_0 < 50$ cm/s. In this case the first term on the left-hand side of Equation (50) can be neglected. Moreover, we can put $w(x,t) = 0$ in (55) to leading order. This means that, if the penetration velocity is small, the flow in region I does not depend on elasticity of the body and is determined by Equations (18)–(20). The deflection of the beam $w(x,t)$ can be found from (50) in the form

$$w(x,t) = \frac{1}{EJ} p(0,t)\psi(x),$$

where

$$\psi(x) = \frac{L^4}{360}\left(61 - 75\frac{x^2}{L^2} + 15\frac{x^4}{L^4} - \frac{x^6}{L^6}\right).$$

Equation (55) provides the outflow velocity $\hat{u}(t)$ as

$$\hat{u}(t) = \frac{1}{h - V_0 t}\left(V_0 L - \int_0^L w_t(x, t)\mathrm{d}x\right).$$

Substituting the approximate formula for the deflection in the last equation, we find

$$\hat{u}(t) = \frac{V_0 L}{h - V_0 t}\left(1 - \frac{233}{1080}\frac{\rho_0 V_0^2 L^6}{EJh^3}(1 - V_0 t/h)^{-3}\right).$$

therefore the flexibility of the body bottom reduces the outflow velocity. The last formula is not valid as $V_0 t/h \to 1$ when problem (50)–(56) has to be considered as coupled for any velocity of the body.

## 9. Three-dimensional impact problem

Impact by a cylinder with flat bottom and an arbitrary cross-section $\Omega$ onto shallow liquid layer is considered within the framework of the scheme described in the previous sections. In region I between the entering body and the bottom of the liquid layer the flow is governed at leading order as $\varepsilon \to 0$ by the equation

$$\nabla \cdot [(h - s(t))\nabla\varphi] = s'(t) \quad ((x, y) \in \Omega) \tag{57}$$

with respect to the velocity potential $\varphi(x, y, t)$, where $\varepsilon = h/L$, $h$ is the thickness of the liquid layer, $L$ is the characteristic linear dimension of the region $\Omega$, equation $z = -s(t)$ describes the position of the body, $\nabla\varphi = (\varphi_x, \varphi_y)$ is the vector of the liquid velocity along the bottom. The pressure $p(x, y, t)$ beneath the body is given by the formula

$$p(x, y, t) = -\rho_0(\varphi_t + \tfrac{1}{2}(\nabla\varphi)^2), \tag{58}$$

which follows from the Cauchy–Lagrange integral at leading order as $\varepsilon \to 0$. In order to match the pressure distributions in regions I and III (see Figure 4), it is required that the pressure is zero at the boundary $\partial\Omega$ of region I. This assumption with account for (58) leads to the boundary condition

$$2\varphi_t + (\nabla\varphi)^2 = 0 \quad ((x, y) \in \partial\Omega). \tag{59}$$

The boundary-value problem (57) and (59) was studied by Yih [4] for both circular and elliptic plates. It was revealed that the velocity of falling plate decays exponentially with time. In order to determine the liquid flow outside the falling body, more information about the outflow velocity is required.

It is clear that Equation (57) admits solutions of the form

$$\varphi(x, y, t) = \frac{s'(t)}{h - s(t)}\Phi(x, y), \tag{60}$$

where the function $\Phi(x, y)$ satisfies Poisson's equation

$$\Delta\Phi = 1 \quad ((x, y) \in \Omega). \tag{61}$$

Substitution of (60) in the boundary condition (59) provides the equation

$$2[s''(t)(h - s(t))(s'(t))^{-2} + 1]\Phi + (\nabla\Phi)^2 = 0 \quad ((x, y) \in \partial\Omega),$$

which can be satisfied if and only if

$$s''(t)[h - s(t)] = k[s'(t)]^2,$$

where $k$ is constant. The equation is solved under the initial conditions $s(0) = 0$, $s'(0) = V_0$. The solution is

$$s(t) = h(1 - [1 - V_0(k+1)t/h]^{\frac{1}{k+1}}). \tag{62}$$

The entry velocity $s'(t)$ is bounded if $-1 < k \leqslant 0$. In particular, $s(t) = V_0 t$ if $k = 0$. Finally, the boundary condition with respect to the function $\Phi(x, y)$ is

$$(\nabla\Phi)^2 + \beta\Phi = 0 \quad ((x, y) \in \partial\Omega), \tag{63}$$

where $\beta = 2(1 + k)$, $\beta > 0$, $k$ is the parameter in formula (62). The pressure $p(x, y, t)$ is given as

$$p(x, y, t) = \frac{\rho_0 (s')^2}{2(h - s)^2} q(x, y), \tag{64}$$

$$q(x, y) = -[\beta\Phi + (\nabla\Phi)^2].$$

The boundary-value problem (61), (63) is not easy to analyse for an arbitrary region $\Omega$. Consider first the axisymmetric case, where $\Omega$ is the circle $x^2 + y^2 < R^2$. In the polar coordinate system $r, \theta$, where $x = r\cos\theta$, $y = r\sin\theta$, the function $\Phi$ is independent of $\theta$ and satisfies

$$\frac{\partial^2\Phi}{\partial r^2} + \frac{1}{r}\frac{\partial\Phi}{\partial r} = 1 \quad (r < R), \tag{65}$$

$$\left(\frac{\partial\Phi}{\partial r}\right)^2 + \beta\Phi = 0 \quad (r = R).$$

We find

$$\Phi(x, y) = \frac{1}{4}r^2 - \frac{1 + \beta}{4\beta}R^2, \tag{66}$$

$$q(x, y) = \tfrac{1}{4}(1 + \beta)(R^2 - r^2),$$

which is in agreement with the results by Yih [4]. The outflow velocity $(\partial\varphi/\partial r)(x, y, t)$, $(x, y) \in \partial\Omega$, is independent of $\theta$ and equal to

$$\frac{\partial\varphi}{\partial r} = \frac{s'(t)R}{2(h - s(t))}.$$

For an elliptic cross-section of the entering cylinder, $\Omega = \{x, y \mid x^2/a^2 + y^2/b^2 \leqslant 1\}$, $b \geqslant a$, the solution of problem (61) and (63) has the form (see [4] and [7])

$$\Phi(x, y) = Ax^2 + By^2 - D. \tag{67}$$

This function satisfies Equations (61) and (63) if the constants $A$, $B$ and $D$ satisfy the system

$$(4A^2 + \beta A)a^2 = \beta D, \qquad (4B^2 + \beta B)b^2 = \beta D, \quad A + B = \tfrac{1}{2}. \tag{68}$$

Formula (64) indicates that the pressure $q(x, y)$ takes its maximum value $q_* = \beta D$ at the centre of the region $\Omega$, $x = 0$, $y = 0$. System (68) yields

$$A = (\sqrt{\beta^2 + 16q^*/a^2} - \beta)/8, \qquad B = (\sqrt{\beta^2 + 16q^*/b^2} - \beta)/8, \tag{69}$$

where

$$q_* = \tfrac{1}{4}a^2 w^{-4}(2+\beta)^2 \left[2 - w^2 - 2\sqrt{1 - w^2 + \beta^2 w^4/[4(2+\beta)^2]}\right], \tag{70}$$

$$w = \sqrt{1 - a^2/b^2}.$$

The solutions (67), (69) and (70) correspond to solution (18) of the plane problem and solution (66) of the axisymmetric problem as $b \to \infty$ and $w \to 0$, respectively.

## 10. Conclusion

The suggested scheme of the liquid flow caused by a box-like structure penetration provides approximate description of the process and makes it possible to predict its peculiarities. It is worth noting that the pressure in the jet root may be higher than the pressure on the bottom of the entering body. Description of the liquid flow outside of region I, which is just beneath the body bottom, is given for the plane case only. Extension of the description onto the general three-dimensional case is straightforward. The reason for that is connected with the fact that the flow in the jet root is approximately two-dimensional.

The shallow-water approximation is very helpful because it not only relates to many practical situations but also provides ideas on peculiarities of the liquid flow under the impact in general case. The problem discussed above was solved numerically by Protopopov [8] for $\varepsilon = 1$. The inclination of the splash jet towards the body, which is mentioned in Section 6, was detected numerically. Comparing the analytical results with those by Protopopov [8], we conclude that the present approach provides the ’sketch’ of the real flow under the impact.

The amount of the liquid piled up during the impact is greater than the amount replaced by the entering body. Owing to the inertia effects a part of the basin bottom around the body will be almost dry at the end of the impact stage. Thereafter the gravity acts to restore the elevated portion of the liquid, which will move both towards the body and away from it. The flow towards the body hits its side wall, and the flow away from the body may form a solitary wave. The main part of the energy of the body motion is dissipated near the body. The portion of the body energy, which is taken away from the body with gravity waves, was estimated in [2] as 0·024 for particular conditions of the experiments.

The blunt-body impact on shallow water was analysed in [1] for the plane case. In the three-dimensional problem the geometry of the contact region (region I) is unknown in advance and has to be found together with the liquid flow. We expect that the exact solutions to shallow-water wave equations found by Thacker [9] will be useful to analyse the impact by an elliptic paraboloid onto shallow water.

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